Adaptive Stopping Criteria for Iterative Solver Applied to Potential Formulations in Magnetostatic Problems

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In this paper, we introduce an adaptive stopping criteria for iterative solver applied to finite element discretizations of both scalar and vector potential formulations in magnetostatic problems. We base our developments on an equilibrated stress a posteriori error estimate distinguishing the different error components, namely the discretization error component and the algebraic solver error component. Our adaptive strategy stops the algebraic iterations when the algebraic error component does not have a significant influence on the discretization error. Numerical examples with analytic solution show the performance of the proposed adaptive strategies.

Index Terms—**Adaptive stopping criteria, a posteriori error estimate, guaranteed bound, interplay between error components**

I. Introduction

Nowadays, the finite element method is widely used to study electromagnetic systems. Unfortunately during the numerical resolution, a large number of CPU time is wasted due to the unnecessary computational cost. For example, in the finite element computation, the total numerical error is the sum of discretization error and algebraic error. When the mesh is constructed, the discretization error cannot be improved, so it is interesting to avoid a large number of iterations in algebraic solver which leads a too small algebraic error. Recently, a stopping criteria based on a posteriori estimate is developed for nonlinear diffusion PDEs in a general framework [1]. With a posteriori estimate, we can distinguish different error components in the numerical resolution, namely the discretization error, algebraic error, and possibly linearization error. This result is extended to the linear Stokes problems, and applied to any iterative solver [2]. In this work, we develop this strategy to the potential formulations in the magnetostatic problems. Implementation into the FreeFem++ programming language is invoked [3]. An academic example is provided to illustrate the performance of our adaptive stopping criteria.

II. Numerical model

Given a divergence-free applied current density $\mathbf{J}_{\mathrm{S}}$, the magnetic flux density $\mathbf{B}$ and magnetic field $\mathbf{H}$ verify the following equation:

$$\mathrm{div}\mathbf{B} = 0; \quad \mathbf{rot}\mathbf{H} = \mathbf{J}_{\mathrm{S}}; \quad \mathbf{B} = \mu\mathbf{H},$$

where μ represents for the magnetic permeability.

Two following potential formulations can be obtained by using the vector potential $\mathbf{A}$ s.t. $\mathbf{B} = \mathbf{rot}\mathbf{A}$ and the scalar potential Ω s.t. $\mathbf{H} = \mathbf{H}_{\mathrm{S}} - \mathbf{grad}\Omega$, where $\mathbf{rot}\mathbf{H}_{\mathrm{S}} = \mathbf{J}_{\mathrm{S}}$:

$$\mathbf{rot}(\frac{1}{\mu}\mathbf{rot}\mathbf{A}) = \mathbf{J}_{\mathrm{S}} \text{ and } \mathrm{div}(\mu\mathbf{grad}\Omega) = \mathrm{div}(\mu\mathbf{H}_{\mathrm{S}}).$$

III. Adaptive stopping criteria

Algorithm 1 classical Iterative algebraic algorithm

1) Chose initial approximations $\mathbf{A}_h^0 \in V_h$ and a tolerance $\varepsilon > 0$.
2) **For** $i = 1 \ldots +\infty$:
   a) Compute $\mathbf{A}_h^i \in V_h$, typically from $\mathbf{A}_h^{i-1}$.
   b) Set up the residual equation, yielding $\mathrm{R}_{\mathbf{A}}^i \in \mathbb{R}^M$ with
   $$(\mu^{-1}\mathbf{rot}\mathbf{A}_h^i, \mathbf{rot}\mathbf{A}'_j) = (\mathbf{J}_{\mathrm{S}}, \mathbf{A}'_j) - \mathrm{R}_{\mathbf{A}}^i, \forall 1 \le j \le M.$$
   c) If $\mathrm{R}_{\mathbf{A}}^i$ is small enough, stop.

   EndFor

Algorithm 2 adaptive iterative algebraic algorithm

1) In step 1, choose also a fixed additional iteration count $\nu_0 > 0$ and real parameters γ_{rem} and $\gamma_{\mathrm{alg}} > 0$, typically of order 10^{-1}. Set $\nu := \nu_0$.
2) On the given iteration i of the algebraic solver in step 2a, consider $\nu := \nu_0$ additional iterations. This gives $\mathbf{A}_h^{i+\nu} \in V_h$ with $\mathrm{R}_{\mathbf{A}}^{i+\nu} \in \mathbb{R}^M$such that
$$(\mu^{-1}\mathbf{rot}\mathbf{A}_h^{i+\nu}, \mathbf{rot}\mathbf{A}'_j) = (\mathbf{J}_{\mathrm{S}}, \mathbf{A}'_j) - \mathrm{R}_{\mathbf{A}}^{i+\nu}, \forall 1 \le j \le M. \quad (1)$$
3) Compute the estimators $\eta_{\mathrm{rem}}, \eta_{\mathrm{alg}}$, . Check the balancing criterion
$$\eta_{\mathrm{rem}} \le \gamma_{\mathrm{rem}}\eta_{\mathrm{alg}}. \quad (2)$$
   If not satisfied, continue performing ν_0 additional iterations and updating $\nu := \nu + \nu_0$, until (2) is satisfied. This again gives (1).
4) On step 2c, compute the estimator η_{disc} and stop the algebraic solver when
$$\eta_{\mathrm{alg}} \le \gamma_{\mathrm{alg}}\eta_{\mathrm{disc}}. \quad (3)$$
   If not satisfied, update $i := i + \nu$.

We consider the vector potential $\mathbf{A}$ formulation as example. Using the finite element method, we consist in find the weak solution $\mathbf{A}_h \in V_h$ such that

$$(\mu^{-1}\mathbf{rot}\mathbf{A}_h, \mathbf{rot}\mathbf{A}') = (\mathbf{J}_{\mathrm{S}}, \mathbf{A}'), \quad \forall \mathbf{A}' \in V_h.$$

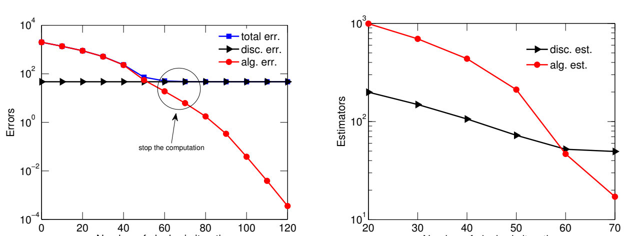
The classical iterative algebraic solver is presented as shown in Algorithm 1. Based on the development of a posteriori estimate and stopping criteria, we propose our iterative algorithm with adaptive stopping criteria shown in Algorithm 2. Recall here $\mathbf{A}$ the exact solution, $\mathbf{A}_h$ the numerical solution with exact algebraic solver, and $\mathbf{A}_h^i$ the numerical solution at $i-th$ algebraic solution, we define here the different error components by:

$$\begin{aligned} \text{Total error:} &= \|\mu^{-1/2}\mathbf{rot}(\mathbf{A} - \mathbf{A}_h^i)\|, \\ \text{Discretization error:} &= \|\mu^{-1/2}\mathbf{rot}(\mathbf{A} - \mathbf{A}_h)\|, \\ \text{Algebraic error:} &= \|\mu^{-1/2}\mathbf{rot}(\mathbf{A}_h - \mathbf{A}_h^i)\|. \end{aligned}$$

The estimators $\eta_{\mathrm{alg}}, \eta_{\mathrm{disc}}, \eta_{\mathrm{rem}}$ are constructed using the flux reconstruction technique on local problems [1], [4], which will be detailed in the full paper.

IV. Numerical application

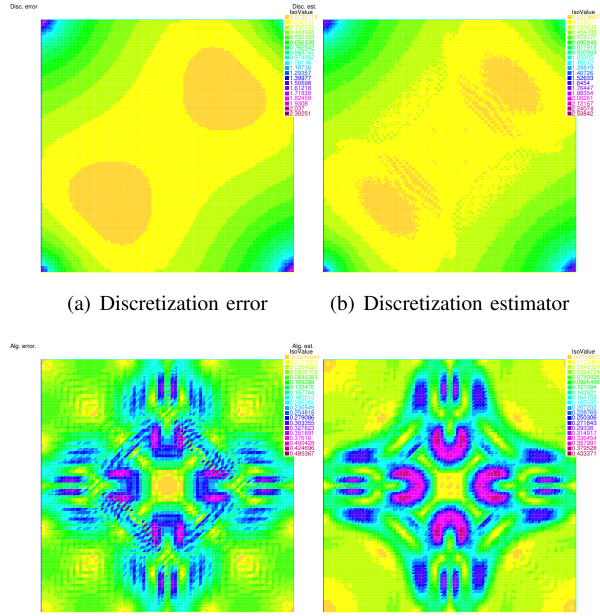
This section presents a numerical assessment of our a posteriori error estimates and of the proposed adaptive stopping criteria. To illustrate our theoretical results, an academic example was studied. Suppose a unit cube crossed by a current density of 10 MA/m [5], the analytic solution for the magnetic flux density is known [6]. We consider here the vector potential $\mathbf{A}$ formulation. Fig. 1 represents the dependence of different error components (total error, discretization error, and algebraic error) and of correspondent estimators (algebraic estimator and discretization estimator) on the algebraic iteration. As the mesh is chosen, the discretization error is fixed. The total error (sum of discretization error and algebraic error) decreases rapidly in Fig. 1(a) for the first 60–70 algebraic iterations and then almost stagnates, since the influence of the algebraic error becomes negligible. This is exactly the point where our adaptive stopping criteria makes as shown in Fig. 1(b), leading to an important economy of the algebraic iterations (70 iterations instead of 120 iterations in total).



(a) Exact errors (b) Estimators

Fig. 1. Different error components and estimators as a function of algebraic iterations.

In Fig. 2, we display the spatial distribution of the two error components on left (discretization error and algebraic error)



(a) Discretization error (b) Discretization estimator

(c) Algebraic error (d) Algebraic estimator

Fig. 2. Spatial distributions of the different error components and corresponding estimates.

and of the corresponding estimators on right (discretization estimator and algebraic estimator). A nice match can be observed. That means using our adaptive stopping criteria and a posteriori estimate, we can avoid an important unnecessary algebraic iteration, at the same time, an accordant error distribution map can be generated to the mesh refinement.

V. Conclusion

Adaptive stopping criteria based on a posteriori estimates for iterative solver is introduced for both $\mathbf{A}$ and Ω potential formulations in magnetostatic problems. An academic example with Freefem++ implementation is carried out to show the performance of our proposed adaptive stopping criteria. With our adaptive strategy, an important number of unnecessary algebraic iterations can be avoided, an accordant error distribution map can be provided to the mesh refinement.

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